Signals and Systems-23EC02

UNIT-III
Analysis of Linear Systems

B.Tech., III-Sem., ECE

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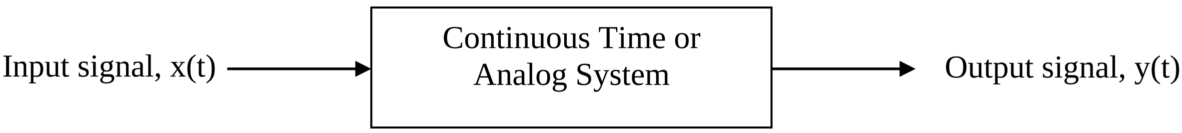
1. Introduction:

- System can be defined as the collection of objects or elements or components and all these things should be interconnected in such a way to achieve an objective or predefined result or outcome.

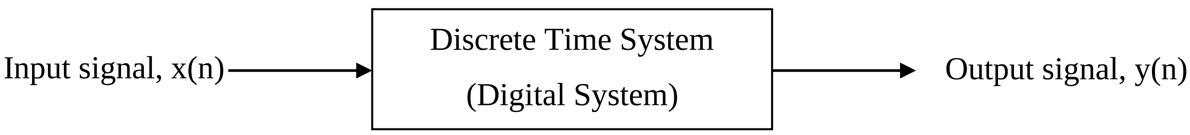
  Examples:
  - ✓ Amplifier
  - ✓ Digital Multimeter (DMM)
  - ✓ Cathode Ray Oscilloscope (CRO)
  - ✓ Regulated Power Supply (RPS)
  - ✓ Function Generator (FG) or Signal Generator (SG)
  - ✓ Mobile Phone
  - ✓ Laptop
  - ✓ Calculator
  - ✓ Central Processing Unit (CPU)

- Based on the type of input applied, components used in the design and type output, systems are classified into two types.
  - ✓ Continuous Time or Analog Systems
  - ✓ Discrete Time and Digital Systems

- Continuous time or analog systems are those for which both the input and output are continuous time signals and are constructed by using analog components, like resistors, capacitors, inductors, diodes, transistors, analog ICs, etc.

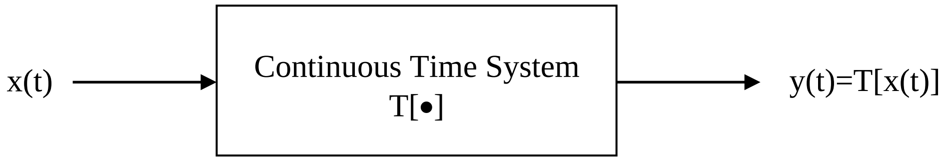


- Discrete time systems are those for which both input and output are discrete time signals and are constructed by using discrete components, like adders, constant multipliers and delays (memories). In the case of digital systems, both input and output are digital signals.



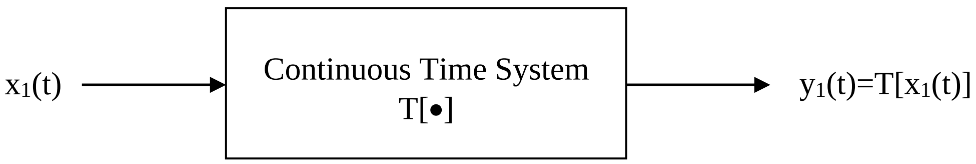
2. Linear and Non-Linear Systems:

Let us consider a continuous time system having input x(t) and output y(t)

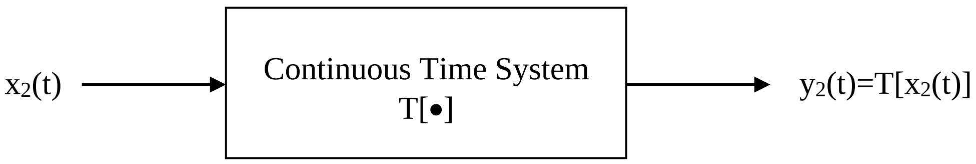


Where T[•] is transform operator and the relation between input x(t) and output y(t) of a continuous time system is represented with y(t) = T[x(t)]. It shows the output y(t) is the transformation of input x(t).

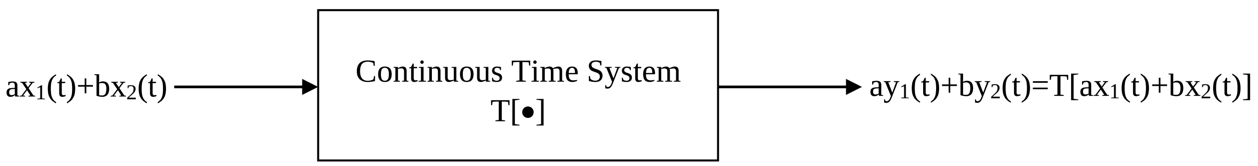
Apply $x_1(t)$ as input to the system T[•] and observe the output, take it as $y_1(t)$



Apply $x_2(t)$ as input to the system T[•] and observe the output, take it as $y_2(t)$



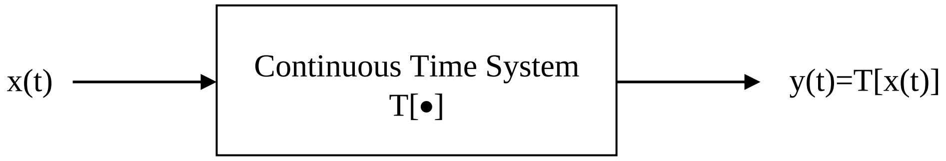
Now apply the linear combination of previous inputs $x_1(t)$ and $x_2(t)$, i.e. $ax_1(t)+bx_2(t)$ as input to the system T[•] and observe the output. If the output is $ay_1(t)+by_2(t)$, then the given system is called linear otherwise the system is nonlinear.



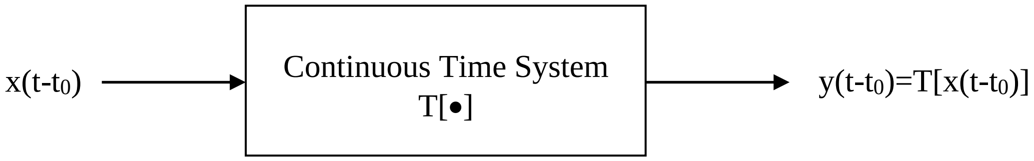
- Condition for a linear system is $ay_1(t)+by_2(t) = T[ax_1(t)+bx_2(t)]$.
- Every linear system must satisfy the superposition principle.
- Linearity is the combination of Additivity and Homogeneity.
- Additivity $\Rightarrow T[x_1(t)+x_2(t)] = T[x_1(t)]+T[x_2(t)] = y_1(t)+y_2(t)$.
- Homogeneity $\Rightarrow T[kx(t)] = k\,T[x(t)] = ky(t)$.

3. Time Invariant and Variant Systems:

Let us consider a continuous time system having input x(t) and output y(t)



- Apply delayed form of x(t), i.e $x(t-t_0)$ as input to the system T[•] and observe the output.
- If the output is $y(t-t_0)$, then the given system is called time invariant.
- Condition for a time invariant system is $y(t-t_0)=T[x(t-t_0)]$.



- If the output is not equal to $y(t-t_0)$, then the given system is time variant. i.e. $y(t-t_0)\neq T[x(t-t_0)]$.

- If a particular system satisfies both Linear property and Time Invariant property, then that system is called Linear Time Invariant (LTI) system.

  Ex: $y(t) = \frac{d}{dt}[x(t)]$

- If a particular system satisfies Linear property and fails to satisfy Time Invariant property, then that system is called Linear Time Variant (LTV) system.

  Ex: $y(t) = x(2t)$

- If a particular system fails to satisfy Linear property and satisfy Time Invariant property, then that system is called Non-Linear Time Invariant (NLTI) system.

  Ex: $y(t) = 2x(t) + 3$

- If a particular system fails to satisfy both Linear property and Time Invariant property, then that system is called Non-Linear Time Variant (NLTV) system.

  Ex: $y(t) = x(2t) + 3$

4. Concept of Convolution in Time and Frequency Domain:

- Convolution is an operation, which is used in almost all signal processing applications to analyze signals and systems in both the time and frequency domain.
- Convolution is a special operation, which includes four different operations, namely
  - ✓ Folding,
  - ✓ Shifting,
  - ✓ Multiplication and
  - ✓ Integration in the case of continues time signals/ Summation in the case of discrete time signals.

- Convolution in continuous time domain can be defined as

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)d\tau$$

  Where, $x_1(t)$ and $x_2(t)$ are two continuous time signals

- If $x_1(t)$, $x_2(t)$ are two continuous time signals and $FT[x_1(t)] = X_1(w)$, $FT[x_2(t)] = X_2(w)$, then $FT[x_1(t) * x_2(t)] = X_1(w)\, X_2(w)$ is called time convolution theorem. i.e, convolution in time domain leads to multiplication in frequency domain.

- If $x_1(t)$, $x_2(t)$ are two continuous time signals and $FT[x_1(t)] = X_1(w)$, $FT[x_2(t)] = X_2(w)$, then $FT[x_1(t)x_2(t)] = \frac{X_1(w)*X_2(w)}{2\pi}$ is called frequency convolution theorem. i.e, convolution in frequency domain leads to multiplication in time domain.

- Procedure to compute the convoluted signal through the graphical method:
  - ✓ First draw the graphical representation of given signals $x_1(\tau)$ and $x_2(\tau)$.
  - ✓ Draw the folding form of $x_2(\tau)$, i.e. $x_2(-\tau)$.
  - ✓ Shift $x_2(-\tau)$ by t units, and draw the graph of $x_2(t-\tau)$.
  - ✓ Take the product of $x_1(\tau)$ and $x_2(t-\tau)$.
  - ✓ Integrate '$x_1(\tau)x_2(t-\tau)$' with respect to τ over wide range of time.

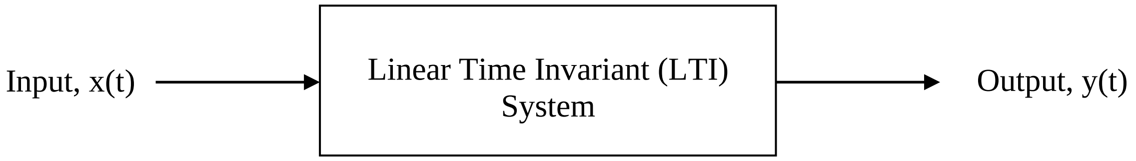
- Convolution in discrete time domain can be defined as

$$x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$

  Where, $x_1(n)$ and $x_2(n)$ are two discrete time signals

5. Response of a LTI Systems:

- Let us consider a Linear Time Invariant (LTI) System having input x(t) and output y(t)



- The ratio between output y(t) to input x(t) in frequency domain representation is called Transfer function or System function or Frequency response of LTI System and it is represented with H(w).

$$H(w) = \frac{FT[y(t)]}{FT[x(t)]} = \frac{Y(w)}{X(w)}$$

- In general, the frequency response H(w) is in complex form and it can be expressed as

$$H(w) = H_R(w) + j\, H_I(w)$$

Where,

$H_R(w)$: Real part of $H(w)$

$H_I(w)$: Imaginary part of $H(w)$

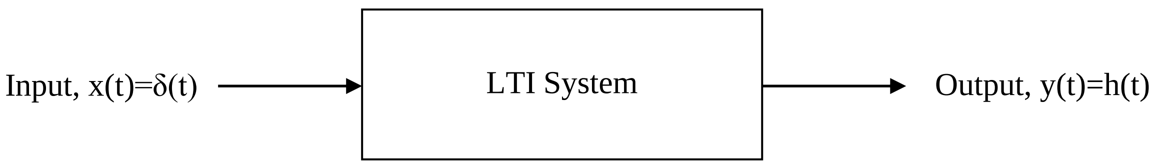
- Magnitude of H(w) is called the magnitude response or Magnitude spectrum, and it can be computed from the formula

$$|H(w)| = \sqrt{[\,H_R(w)\,]^2 + [\,H_I(w)\,]^2}$$

- Phase of H(w) is called the phase response or Phase spectrum, and it can be computed from the formula

$$\angle H(w) = Tan^{-1}\left(\frac{H_I(w)}{H_R(w)}\right)$$

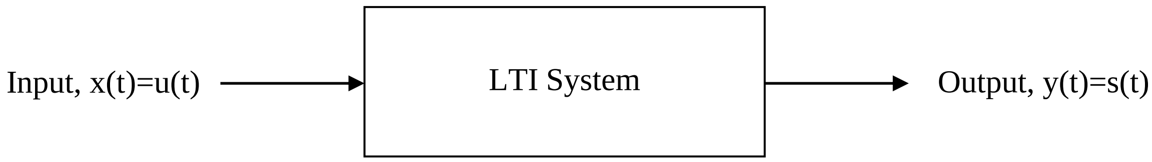
- Response of a LTI system with an input of impulse signal is called is called impulse response and it is represented with h(t).



- Impulse response h(t) can be obtained from the frequency response H(w) by using Inverse Fourier Transform (IFT).

$$h(t) = IFT[H(w)]$$

- Response of a LTI system with an input of step signal is called step response and it is represented with s(t).



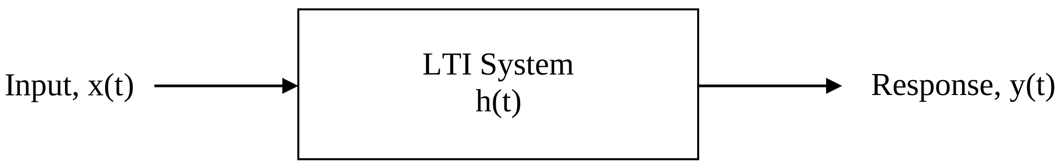
- Relation between impulse response h(t) and step response s(t) of a LTI system:

$We\ know\ that,\ \delta(t) = \frac{d}{dt}u(t)$

$$\Rightarrow h(t) = \frac{d}{dt}s(t)$$

- Response of a LTI system is the convolution of input x(t) and impulse response h(t).

$$y(t) = x(t) * h(t)$$



Proof:

$We\ know\ that,$

$$H(w) = \frac{FT[y(t)]}{FT[x(t)]} = \frac{Y(w)}{X(w)}$$

$$\Rightarrow Y(w) = X(w)H(w)$$

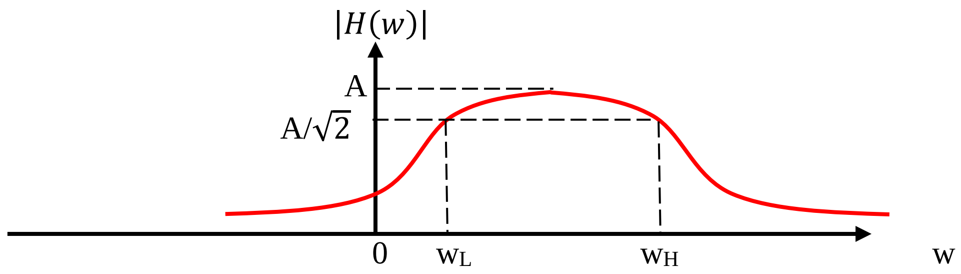
$$\Rightarrow IFT[Y(w)] = IFT[X(w)H(w)]$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$i.e.\ y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)d\tau$$

6. System Bandwidth, Signal Bandwidth and Rise Time

- The range of frequencies for which the magnitude response $|H(w)|$ of the system remains within $\frac{1}{\sqrt{2}}$ of its maximum amplitude is called Bandwidth of a system.



Where,

A : Maximum amplitude of magnitude response $|H(w)|$

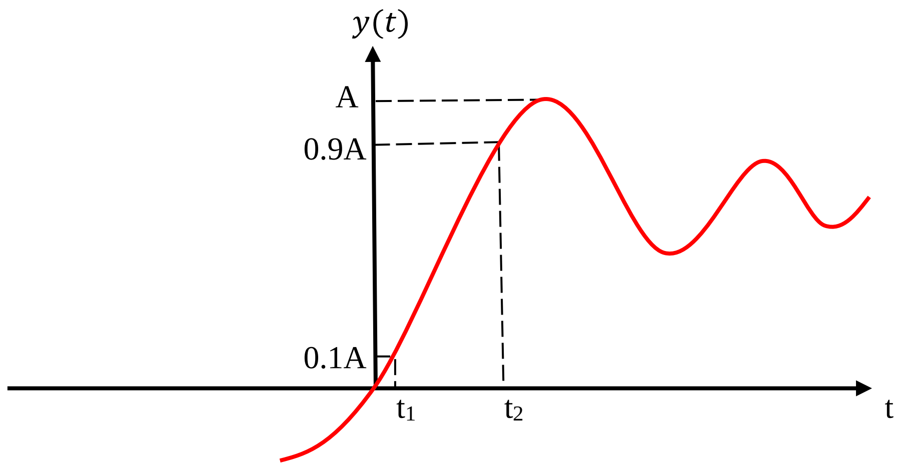
w_L : Lower cutoff frequency in rad/sec

w_H : Upper cutoff frequency in rad/sec

- Bandwidth is the difference between upper cutoff frequency and lower cutoff frequency.

$$Bandwidth\ (BW) = \begin{cases} w_H - w_L, & w_L > 0 \\ w_H, & w_L < 0 \end{cases}$$

- Similarly, for any signal h(t), the bandwidth can be computed through above procedure.
- The time required for the response y(t) to rise from 10% to 90% of its final value is called Rise Time.

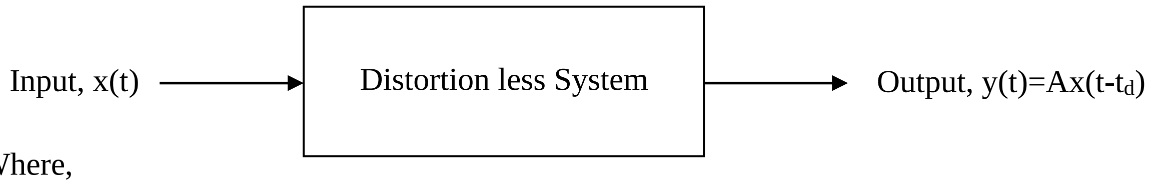


$$Rise\ Time\ (t_r) = t_2 - t_1$$

- Bandwidth is inversely proportional to Rise time, and the product of Bandwidth and Rise time is constant.

7. Distortion less Transmission through a System

- Signal transmission through a system is said to be distortion less only when the output wave shape is exact replica of input wave shape.



Where,

A : Gain of the system

t_d : Delay between output y(t) and input x(t).

- Relation between the output y(t) and input x(t) of a distortion less transmission system

$$y(t) = Ax(t - t_d)$$

- Frequency domain of a distortion less transmission system can be obtained by applying Fourier Transform both side to $y(t) = Ax(t - t_d)$

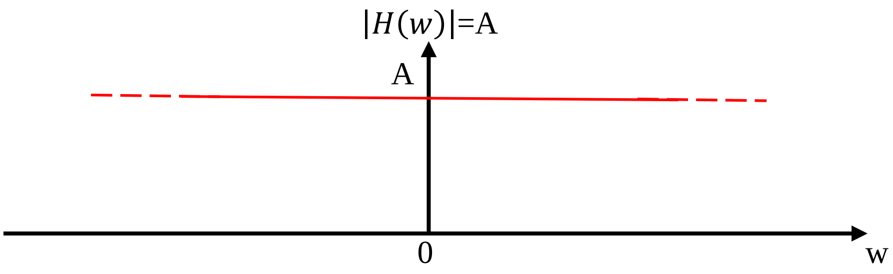
$\Rightarrow FT[y(t)] = FT[Ax(t - t_d)]$

$\Rightarrow Y(w) = Ae^{-jwt_d}X(w)$

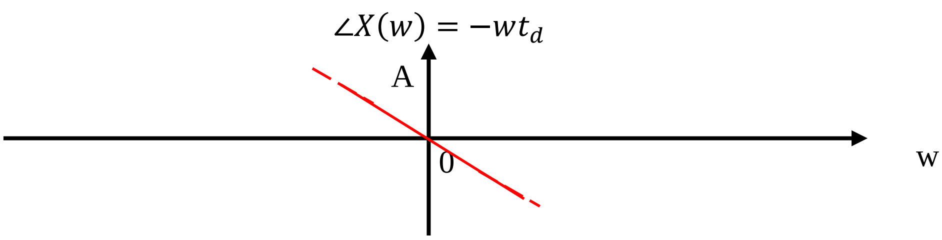
$\Rightarrow \frac{Y(w)}{X(w)} = Ae^{-jwt_d}$

$\Rightarrow H(w) = Ae^{-jwt_d}$

- Magnitude response of a distortion less system is constant, and bandwidth is infinity



- Phase response of a distortion less system is linear



8. Filter Characteristics of Linear Systems

- In general, filters are used to extract required information and attenuate all other unwanted things.

- In communication, filter can be defined as a frequency selective device for various frequencies of signals.

- Characteristics of filter are
  - ✓ Some frequency components may be boosted in strength.
  - ✓ Some frequency components may be attenuated.
  - ✓ Some frequency components may be unchanged.

- For a distortion less filter, the output wave shape is the exact replica of the input wave shape over specified band of frequency signals and attenuates all other unwanted frequency signals.

- Based on frequency response, filters are classified into four types.
  - ✓ Low Pass Filters (LPF)
  - ✓ High Pass Filters (HPF)
  - ✓ Band Pass Filters (BPF)
  - ✓ Band Stop Filters (BSF)

- **Low pass filter** allows only low frequency signals and attenuate all other high frequency signals.

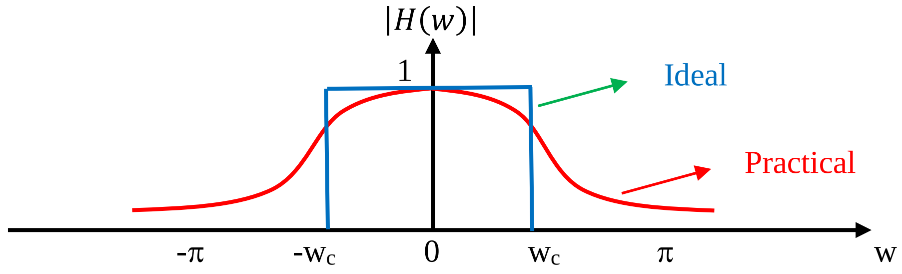
- **High pass filter** allows only high frequency signals and attenuate all other low frequency signals.

- **Band pass filter** allows only a certain band of frequency signals and attenuate all other unwanted band of frequency signals.

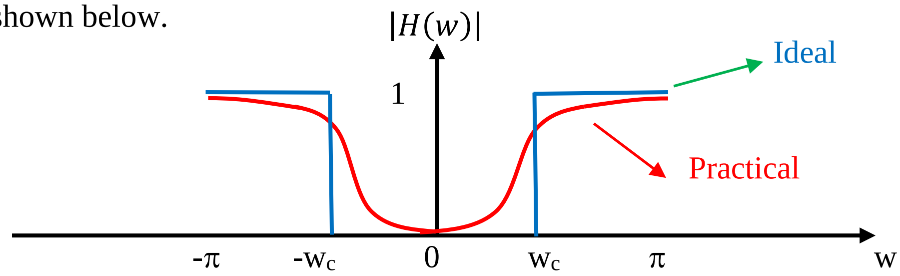
- **Band stop filter** or **Band reject filter** or **Band elimination** filter allows entire band of frequency signals and attenuate a narrow band or unwanted band of frequency signals.

9. Ideal and Practical Characteristics of LPF, HPF, BPF & BSF

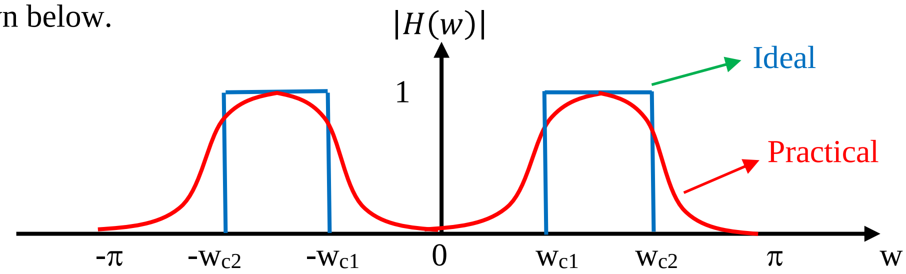
- Low pass filter allows only low frequency signals over the range $-w_C \leq w \leq w_C$ and attenuate all other high frequency signals. Ideal and practical characteristics of a low pass filter as shown below.



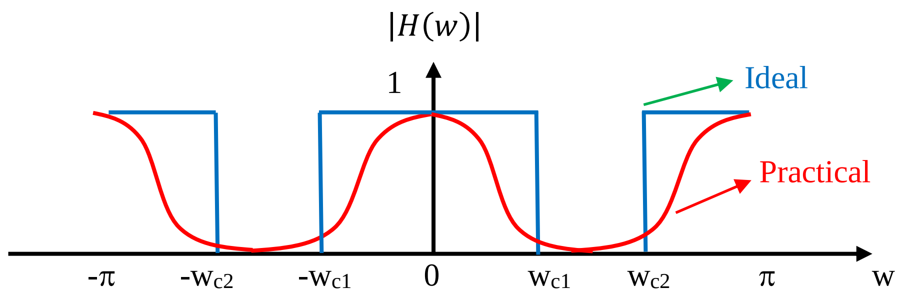
- High pass filter allows only high frequency signals over the range $-\pi \leq w \leq -w_C$ and $w_C \leq w \leq \pi$ and attenuate all other low frequency signals. Ideal and practical characteristics of a high pass filter as shown below.



- Band pass filter allows only a certain band of frequency signals over the range $-w_{C2} \leq w \leq -w_{C1}$ and attenuate all other frequency signals. Ideal and practical characteristics of a band pass filter as shown below.



- Band stop filter allows entire band of frequency signals over the range $-\pi \leq w \leq -w_{C2}$ & $-w_{C1} \leq w \leq w_{C1}$ & $w_{C2} \leq w \leq \pi$.except unwanted band of frequency signals. Ideal and practical characteristics of a band stop filter as shown below.



10. Causality and Poly-Wiener Criterion for Physical Realization

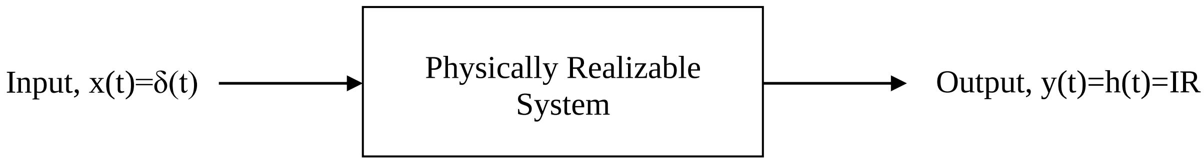
- Causal systems are those for which the present output y(t) depends on only present input x(t) and/or past inputs x(t-1), x(t-2),…. and/or past outputs y(t-1), y(t-2),… but does not depends on future inputs x(t+1), x(t+2),…. and/or future outputs y(t+1), y(t+2),……..

- All static systems are causal, where present output depends only on present inputs. But causal systems may be static or dynamic.

- Non-causal systems are those for which the present output y(t) may depends on future inputs x(t+1), x(t+2),…. and/or future outputs y(t+1), y(t+2),……. All non causal systems are dynamic.

- A physically realizable system can't have a response before the driving function is applied.

- Impulse response of a physically realizable system is zero for t < 0. i.e h(t) = 0, t < 0.



- Physically realizable systems are causal and practical

- A system having system function H(w) is said to be physically realizable if and only if the squared magnitude spectrum is absolutely integrable.

$$\int_{-\infty}^{\infty} |H(w)|^2 dw < \infty$$

- Necessary and sufficient condition for a physically realizable system is known as poly-wiener criterion.

$$\int_{-\infty}^{\infty} \frac{\left|ln|H(w)|\right|}{1+w^2} dw < \infty$$

11. Solved Problems:

(1) Test the continuous time system, $y(t) = \frac{d}{dt}[x(t)]$ for linearity.

Let us consider a continuous time system having input x(t) and output y(t) as shown

$x(t) \longrightarrow$ T[•] $\longrightarrow y(t) = T[x(t)] = \frac{d}{dt}[x(t)]$

Apply $x_1(t)$ as input to the system T[•] and observe the output, take it as $y_1(t)$

$x_1(t) \longrightarrow$ T[•] $\longrightarrow y_1(t) = T[x_1(t)] = \frac{d}{dt}[x_1(t)]$

Apply $x_2(t)$ as input to the system T[•] and observe the output, take it as $y_2(t)$

$x_2(t) \longrightarrow$ T[•] $\longrightarrow y_2(t) = T[x_2(t)] = \frac{d}{dt}[x_2(t)]$

Now apply the linear combination of previous inputs $x_1(t)$ and $x_2(t)$, i.e. $ax_1(t)+bx_2(t)$ as input to the system T[•] and observe the output.

$ax_1(t) + bx_2(t) \longrightarrow$ T[•] $\longrightarrow T[ax_1(t) + bx_2(t)] = \frac{d}{dt}[ax_1(t) + bx_2(t)]$

$$\Rightarrow T[ax_1(t) + bx_2(t)] = a\frac{d}{dt}[x_1(t)] + b\frac{d}{dt}[x_2(t)] - - - - - (1)$$

Determine, $ay_1(t)+by_2(t)$

$$\Rightarrow ay_1(t) + by_2(t) = a\frac{d}{dt}[x_1(t)] + b\frac{d}{dt}[x_2(t)] - - - - - - (2)$$

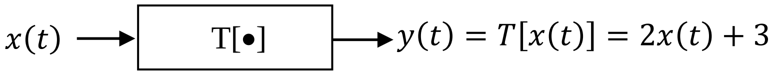
Compare equations (1) and (2)

$$\Rightarrow ay_1(t) + by_2(t)] = T[ax_1(t) + bx_2(t)]$$

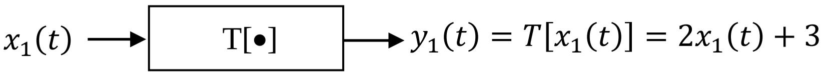
Given system is linear

(2) Test the continuous time system, $y(t) = 2x(t) + 3$ for linearity.

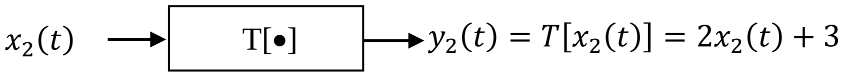
Let us consider a continuous time system having input x(t) and output y(t) as shown



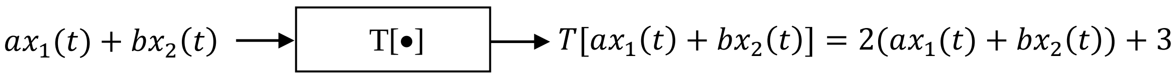
Apply $x_1(t)$ as input to the system T[•] and observe the output, take it as $y_1(t)$



Apply $x_2(t)$ as input to the system T[•] and observe the output, take it as $y_2(t)$



Now apply the linear combination of previous inputs $x_1(t)$ and $x_2(t)$, i.e. $ax_1(t)+bx_2(t)$ as input to the system T[•] and observe the output.



$$\Rightarrow T[ax_1(t) + bx_2(t)] = 2(ax_1(t) + bx_2(t)) + 3$$
$$\Rightarrow T[ax_1(t) + bx_2(t)] = 2ax_1(t) + 2bx_2(t) + 3 - - - - - (1)$$

Apply $ay_1(t)+by_2(t)$

$$\Rightarrow ay_1(t) + by_2(t) = a(2x_1(t) + 3) + b(2x_2(t) + 3)$$
$$\Rightarrow ay_1(t) + by_2(t) = 2ax_1(t) + 3a + 2bx_2(t) + 3b$$
$$\Rightarrow ay_1(t) + by_2(t)] = 2ax_1(t) + 2bx_2(t) + 3a + 3b - - - (2)$$

Compare equations (1) and (2)

$$\Rightarrow ay_1(t) + by_2(t)] \neq T[ax_1(t) + bx_2(t)]$$

Given system is nonlinear

(3) Test the continuous time system, $y(t) = \frac{d}{dt}[x(t)]$ for time invariance.

Let us consider a continuous time system having input x(t) and output y(t) as shown

$$x(t) \longrightarrow \boxed{T[\bullet]} \longrightarrow y(t) = T[x(t)] = \frac{d}{dt}[x(t)]$$

Apply delayed form of x(t), i.e x(t-t_0) as input to the system T[•] and observe the output.

$$x(t-t_0) \longrightarrow \boxed{T[\bullet]} \longrightarrow T[x(t-t_0)] = \frac{d}{dt}[x(t-t_0)] --- (1)$$

$$Given, y(t) = \frac{d}{dt}[x(t)]$$

$$Replace\ 't'\ with\ t-t_0 \Rightarrow y(t-t_0) = \frac{d}{dt}[x(t-t_0)] --- (2)$$

Compare equations (1) and (2)

$$\Rightarrow y(t-t_0) = T[x(t-t_0)]$$

Given system is time invariant.

Note: Differentiator is a Linear Time Invariant (LTI) System.

(4) Test the continuous time system, $y(t) = tx(t)$ for time invariance.

Let us consider a continuous time system having input x(t) and output y(t) as shown

$$x(t) \longrightarrow \boxed{T[\bullet]} \longrightarrow y(t) = T[x(t)] = tx(t)$$

Apply delayed form of x(t), i.e x(t-t_0) as input to the system T[•] and observe the output.

$$x(t-t_0) \longrightarrow \boxed{T[\bullet]} \longrightarrow T[x(t-t_0)] = tx(t-t_0) --- (1)$$

Given, $y(t) = tx(t)$

$$Replace\ 't'\ with\ t-t_0 \Rightarrow y(t-t_0) = (t-t_0)x(t-t_0) --- (2)$$

Compare equations (1) and (2)

$$\Rightarrow y(t-t_0) \neq T[x(t-t_0)]$$

Given system is time variant.

Note: Given system is a Linear Time Variant (LTV) System.

(5) Test the continuous time system, $y(t) = x(2t)$ for time invariance.

Let us consider a continuous time system having input x(t) and output y(t) as shown

$$x(t) \longrightarrow \boxed{T[\bullet]} \longrightarrow y(t) = T[x(t)] = x(2t)$$

Apply delayed form of x(t), i.e x(t-t_0) as input to the system T[•] and observe the output.

$$x(t-t_0) \longrightarrow \boxed{T[\bullet]} \longrightarrow T[x(t-t_0)] = x(2t-t_0) --- (1)$$

Given, $y(t) = x(2t)$

$Replace\ 't'\ with\ t-t_0 \Rightarrow y(t-t_0) = x(2(t-t_0))$

$$\Rightarrow y(t-t_0) = x(2t-2t_0) --- (2)$$

Compare equations (1) and (2)

$$\Rightarrow y(t-t_0) \neq T[x(t-t_0)]$$

Given system is time variant.

Note: Given system is a Linear Time Variant (LTV) System.

(6) Test the continuous time system, $y(t) = x(-t)$ for time invariance.

Let us consider a continuous time system having input x(t) and output y(t) as shown

$$x(t) \longrightarrow \boxed{T[\bullet]} \longrightarrow y(t) = T[x(t)] = x(-t)$$

Apply delayed form of x(t), i.e x(t-t_0) as input to the system T[•] and observe the output.

$$x(t-t_0) \longrightarrow \boxed{T[\bullet]} \longrightarrow T[x(t-t_0)] = x(-t-t_0) --- (1)$$

Given, $y(t) = x(-t)$

$Replace\ 't'\ with\ t-t_0 \Rightarrow y(t-t_0) = x(-(t-t_0))$

$$\Rightarrow y(t-t_0) = x(-t+t_0) --- (2)$$

Compare equations (1) and (2)

$$\Rightarrow y(t-t_0) \neq T[x(t-t_0)]$$

Given system is time variant.

Note: Given system is a Linear Time Variant (LTV) System.

(7) Determine the convoluted signal x(t)*y(t) for (i) a ≠ b (ii) a = b.

Given x(t) = e^{-at} u(t) and y(t) = e^{-bt} u(t), where a and b are positive real numbers.

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-bt}e^{(b-a)\tau}u(\tau)u(t-\tau)d\tau;\ u(\tau) = 1, \tau > 0; u(t-\tau) = 1, t-\tau > 0, \tau < t$$

$$= e^{-bt}\int_{0}^{t} e^{(b-a)\tau}d\tau; u(t)u(t-\tau) = 1, 0 < \tau < t$$

Case 1: If a ≠ b

$$x(t) * y(t) = e^{-bt}\frac{e^{(b-a)\tau}}{b-a}\Big|_0^t$$

$$= e^{-bt}\frac{e^{(b-a)t} - 1}{b-a}$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$

$$\boxed{e^{-at}u(t) * e^{-bt}u(t) = \frac{e^{-at} - e^{-bt}}{b-a}u(t)}$$

Case 2: If a = b

$$x(t) * y(t) = e^{-bt}\int_{0}^{t} e^{(b-a)\tau}d\tau$$

$$= e^{-bt}\int_{0}^{t} e^{0}d\tau$$

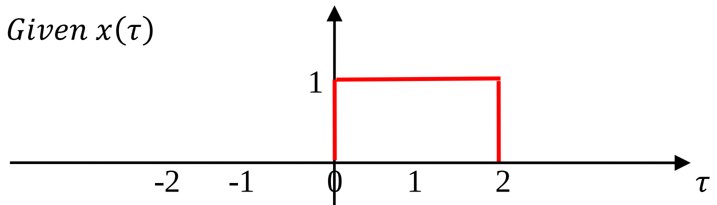
$$= e^{-bt}\int_{0}^{t} d\tau$$

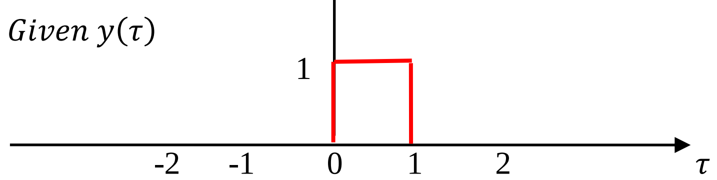
$$= e^{-bt}t$$

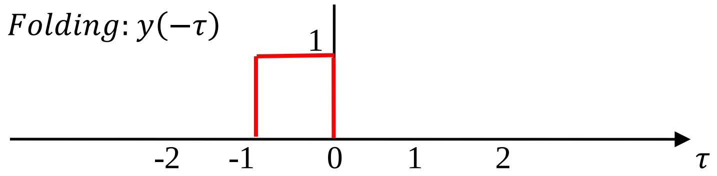
$$\boxed{e^{-at}u(t) * e^{-at}u(t) = te^{-at}u(t)}$$

(8) Evaluate the convoluted signal x(t)*y(t), given x(t)=u(t)-u(t-2) and y(t)=u(t)-u(t-1).

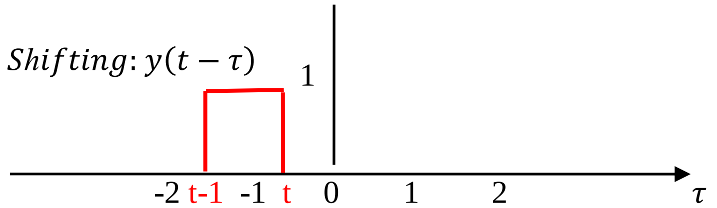
$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$$



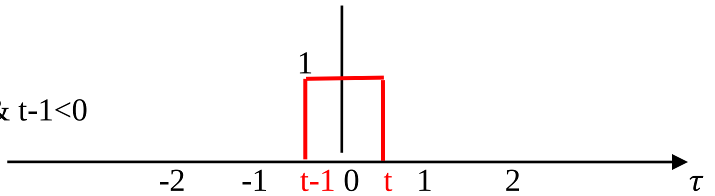




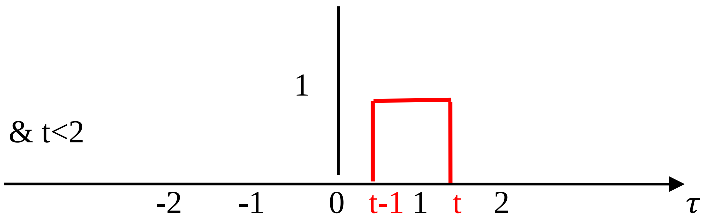
$Case-1$: t<0



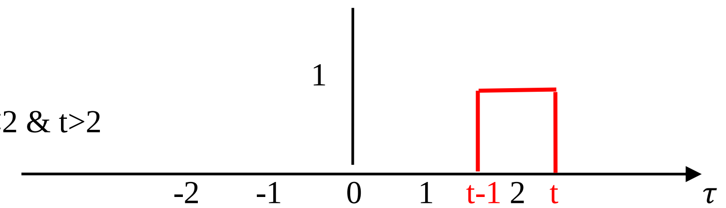
$Case-2$: t>0 & t-1<0



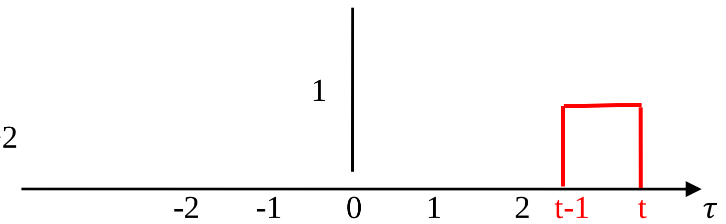
$Case-3$: t-1>0 & t<2



$Case-4$: t-1<2 & t>2



$Case-5$: t-1>2



$Case-1$: t < 0:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = 0, t < 0$$

$Case-2$: t>0 & t-1<0

$$x(t) * y(t) = \int_{0}^{t} 1 d\tau = t, 0 < \ t < 1$$

$Case-3$: t-1>0 & t<2

$$x(t) * y(t) = \int_{t-1}^{t} 1 d\tau = t-(t-1) = 1, 1 < \ t < 2$$

$Case-4$: t-1<2 & t>2

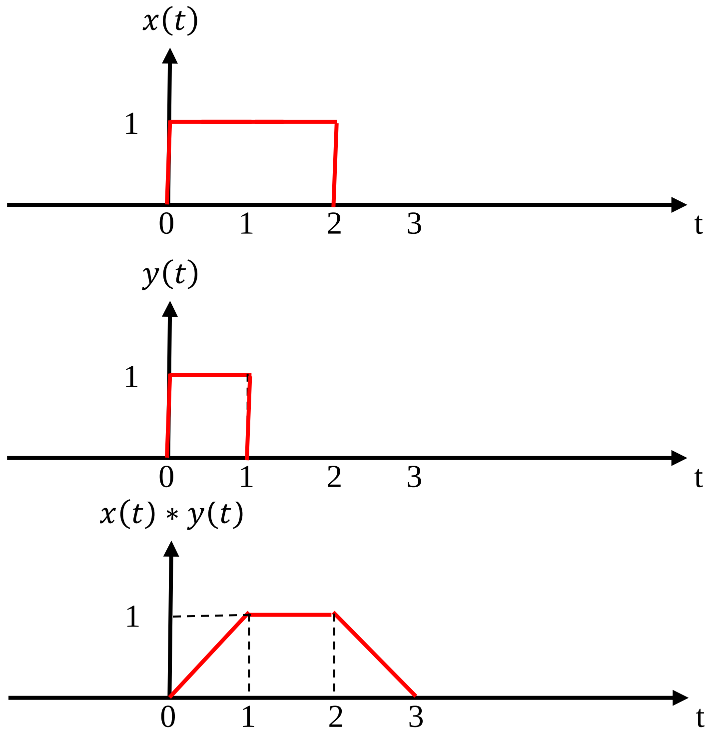
$$x(t) * y(t) = \int_{t-1}^{2} 1 d\tau = 2-(t-1) = 3-t, 2 < \ t < 3$$

$Case-5$: t-1>2

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau = 0, t > 3$$

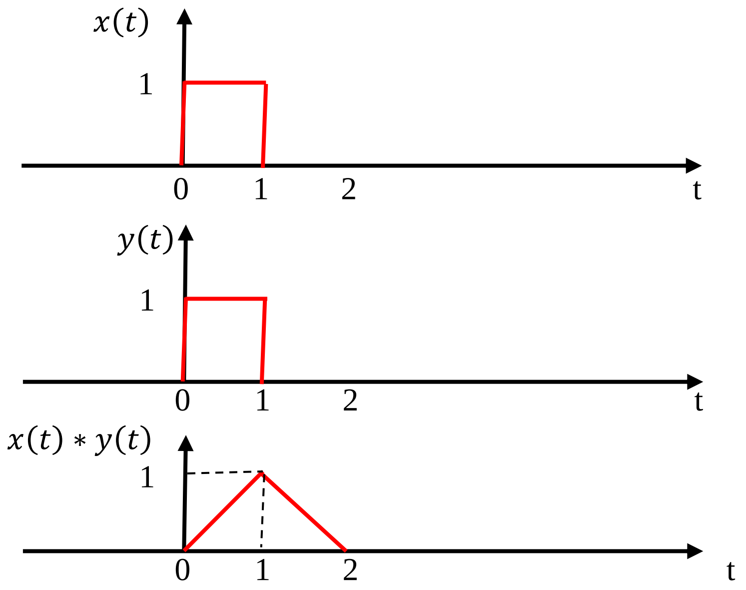
Convoluted Signal

$$x(t) * y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & t > 3 \end{cases}$$



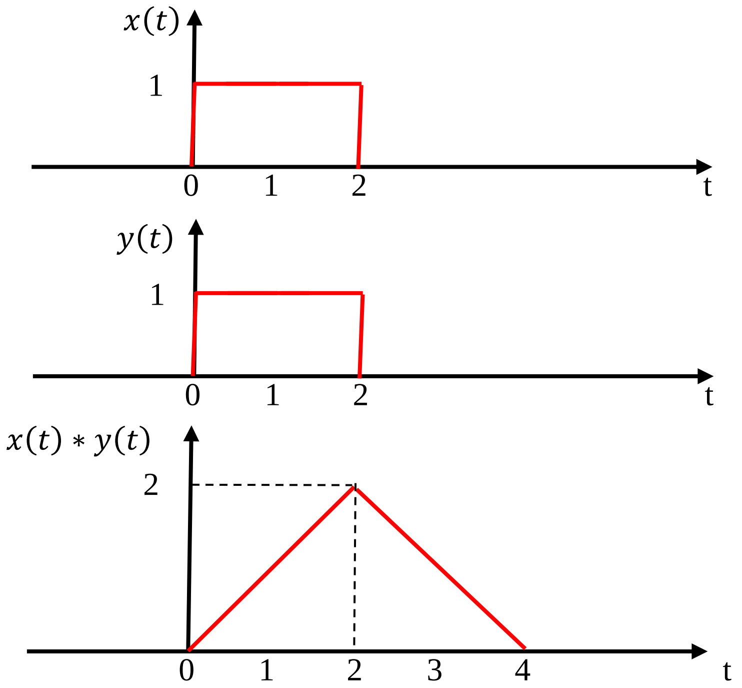
(9) Evaluate the convoluted signal x(t)*y(t), given

(a)x(t)=y(t)=u(t)-u(t-1)



$$x(t) * y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

(b)x(t)=y(t)=u(t)-u(t-2)



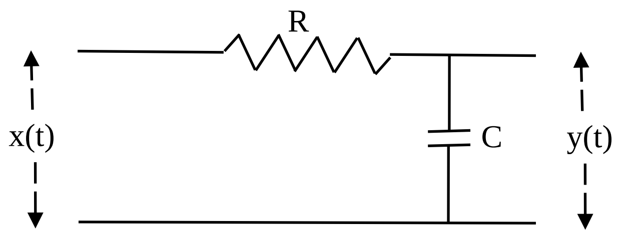
$$x(t) * y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t < 2 \\ 4 - t, & 2 < t < 4 \\ 0, & t > 4 \end{cases}$$

(10) Draw a simple RC low pass filter circuit and determine

(a) Transfer Function, Magnitude Response and Phase Response

(b) Impulse Response and Step Response

(c) Response of a System for an input of $x(t) = e^{-\frac{2}{RC}t}\, u(t)$



(a) Transfer Function

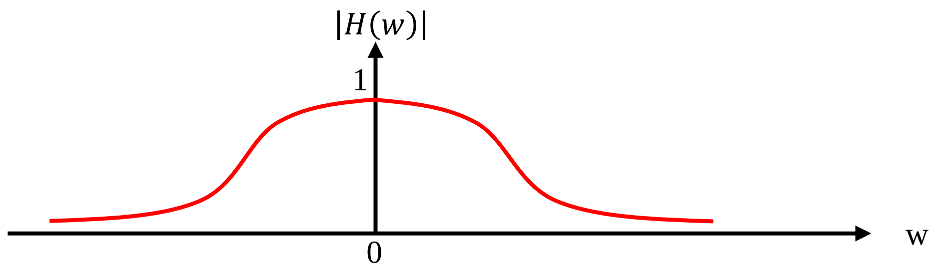
$$H(w) = \frac{Y(w)}{X(w)}$$

$$= \frac{\frac{1}{jwC}}{R + \frac{1}{jwC}}$$

$$= \frac{1}{1 + jwRC}$$

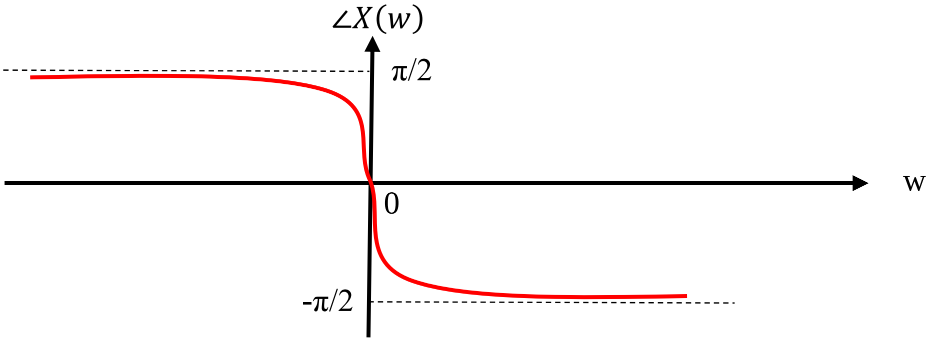
Magnitude Response

$$|H(w)| = \frac{1}{\sqrt{1 + (wRC)^2}}$$



Phase Response

$$\angle X(w) = -\tan^{-1}(wRC)$$



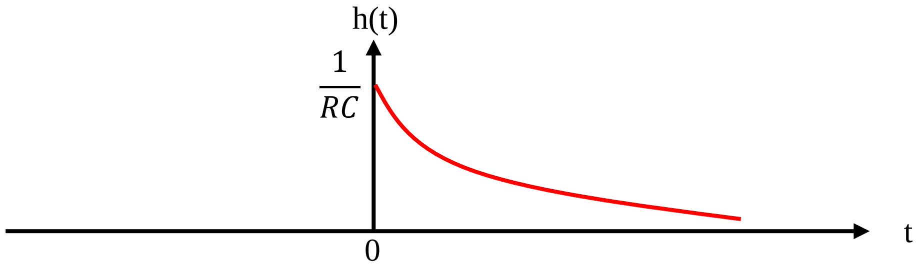
(b) Impulse Response

$$h(t) = IFT[H(w)]$$

$$= IFT\left[\frac{1}{1+jwRC}\right]$$

$$= \frac{1}{RC} IFT\left[\frac{1}{1/RC+jw}\right]$$

$$= \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$



Step Response

$If\ x(t) = u(t), then\ y(t) = s(t) = Step\ Response$

$We\ know\ that,\ \delta(t) = \frac{d}{dt}u(t)$

$$\Rightarrow h(t) = \frac{d}{dt}s(t)$$

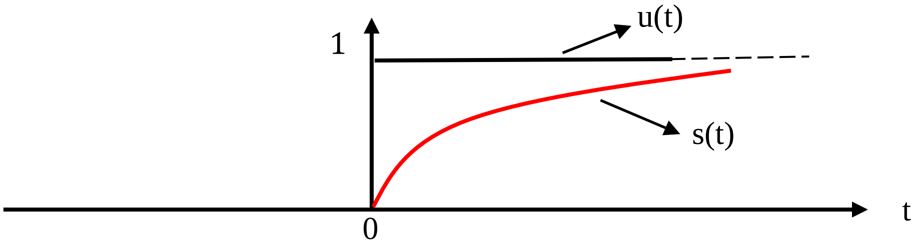
$$\Rightarrow s(t) = \int_0^t h(t)dt$$

$$= \int_0^t \frac{1}{RC} e^{-\frac{t}{RC}} dt$$

$$= \frac{1}{RC} \left. \frac{e^{-\frac{t}{RC}}}{-\frac{1}{RC}} \right|_0^t$$

$$= \frac{e^{-\frac{t}{RC}} - 1}{-1}$$

$$= \left(1 - e^{-\frac{1}{RC}}\right) u(t)$$



(c) Response of the system for an input of $x(t) = e^{-\frac{2}{RC}t}\, u(t)$

$$System\ function, H(w) = \frac{Y(w)}{X(w)}$$

$$\Rightarrow Y(w) = H(w)X(w)$$

$$= \left(\frac{1}{1 + jwRC}\right)\left(\frac{1}{\frac{2}{RC} + jw}\right)$$

$$= \frac{1}{RC} \frac{1}{\left(\frac{1}{RC} + jw\right)\left(\frac{2}{RC} + jw\right)}$$

$$= \frac{1}{\frac{1}{RC} + jw} - \frac{1}{\frac{2}{RC} + jw}$$

$$y(t) = e^{-\frac{1}{RC}t}\, u(t) - e^{-\frac{2}{RC}t}\, u(t)$$

$$= \left(e^{-\frac{1}{RC}t} - e^{-\frac{2}{RC}t}\right) u(t)$$

Process of Evaluating Maxima

$$\frac{d}{dt}\left(e^{-\frac{1}{RC}t} - e^{-\frac{2}{RC}t}\right) = 0$$

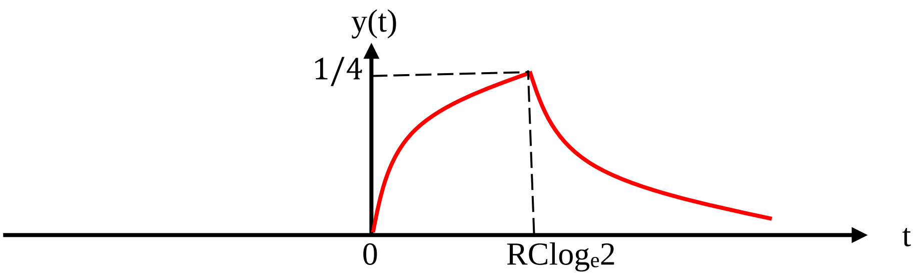
$$\Rightarrow -\frac{1}{RC} e^{-\frac{1}{RC}t} + \frac{2}{RC} e^{-\frac{2}{RC}t} = 0$$

$$\Rightarrow \frac{1}{RC} e^{-\frac{1}{RC}t} = \frac{2}{RC} e^{-\frac{2}{RC}t}$$

$$\Rightarrow e^{\frac{1}{RC}t} = 2$$

$$\Rightarrow t = RC\log_e 2$$

$$\Rightarrow y(t)_{max} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

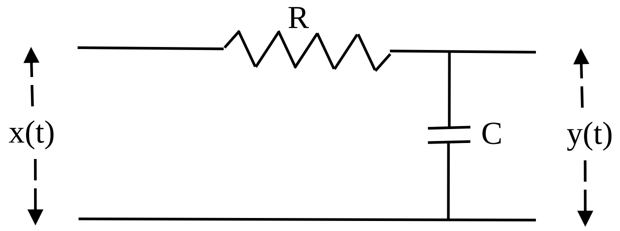


(11) Draw a simple RC low pass filter circuit and determine

(a) Bandwidth (BW)

(b) Rise Time (t_r)

(c) Relation between BW and t_r



(a) Bandwidth (BW)

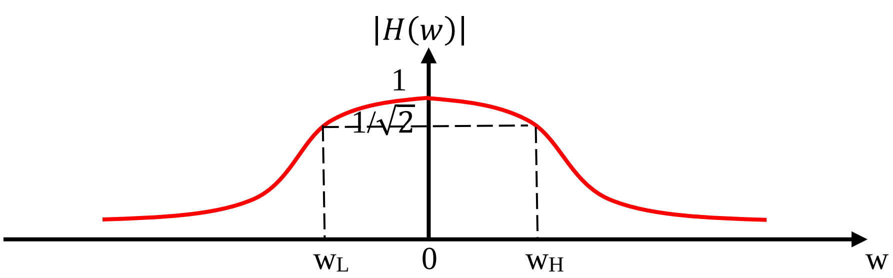
First compute, Frequency Response and Magnitude Response

$$H(w) = \frac{Y(w)}{X(w)}$$

$$= \frac{\frac{1}{jwC}}{R + \frac{1}{jwC}}$$

$$= \frac{1}{1 + jwRC}$$

$$|H(w)| = \frac{1}{\sqrt{1+(wRC)^2}}$$



$$Bandwidth\ (BW) = \begin{cases} w_H - w_L, & w_L > 0 \\ w_H, & w_L < 0 \end{cases}$$

$$\backslash w = w_H \Rightarrow |H(w_H)| = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{1 + (w_H RC)^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 1 + (w_H RC)^2 = 2$$

$$\Rightarrow (w_H RC)^2 = 1$$

$$\Rightarrow w_H RC = \pm 1$$

$$\Rightarrow w_H = \frac{1}{RC} \; and \; w_L = -\frac{1}{RC}$$

$$\Rightarrow Bandwidth\,(BW) = w_H = \frac{1}{RC}\, rad/sec = \frac{1}{2\pi RC} Hz$$

(b) Rise Time

$If\ x(t) = u(t), then\ y(t) = s(t) = Step\ Response$

$$We\ know\ that,\ \delta(t) = \frac{d}{dt}u(t)$$

$$\Rightarrow h(t) = \frac{d}{dt}s(t)$$

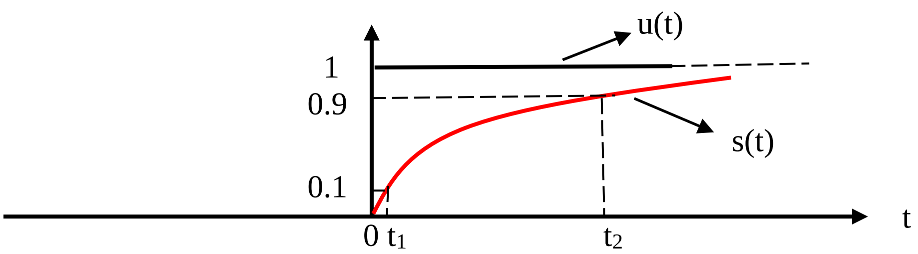
$$\Rightarrow s(t) = \int_0^t h(t)dt$$

$$= \int_0^t \frac{1}{RC} e^{-\frac{t}{RC}}\, dt$$

$$= \frac{1}{RC} \frac{e^{-\frac{t}{RC}}}{-\frac{1}{RC}}\Bigg|_0^t$$

$$= \frac{e^{-\frac{t}{RC}} - 1}{-1}$$

$$= \left(1 - e^{-\frac{1}{RC}t}\right) u(t)$$



$$t = t_1 \Rightarrow s(t_1) = 0.1$$

$$\Rightarrow 1 - e^{-\frac{1}{RC}t_1} = 0.1$$

$$\Rightarrow e^{-\frac{1}{RC}t_1} = 0.9 ------(1)$$

$$t = t_2 \Rightarrow s(t_2) = 0.9$$

$$\Rightarrow 1 - e^{-\frac{1}{RC}t_2} = 0.9$$

$$\Rightarrow e^{-\frac{1}{RC}t_2} = 0.1 ------(2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{e^{-\frac{1}{RC}t_1}}{e^{-\frac{1}{RC}t_2}} = \frac{0.9}{0.1}$$

$$\Rightarrow e^{\frac{t_2-t_1}{RC}} = 9$$

$$\Rightarrow \frac{t_2 - t_1}{RC} loge = log9$$

$$\Rightarrow \frac{t_2 - t_1}{RC} = log_e 9$$

$$\Rightarrow t_2 - t_1 = RClog_e 9$$

$$\Rightarrow t_r = t_2 - t_1 = 2.197RC\ sec$$

(c) Relation between BW and t_r

$$Bandwidth\ (BW) = \frac{1}{2\pi RC} Hz$$

Rise Time $(t_r) = 2.197RC\ sec$

$$BWt_r = \frac{1}{2\pi RC} \times 2.197RC = 0.35$$

$$BWt_r = 0.35$$

or

$$BW = \frac{0.35}{t_r}$$

or

$$t_r = \frac{0.35}{BW}$$

12. Assignment Questions

(1) Test the following continuous time systems for linearity.

(a) $y(t) = 2\frac{d}{dt}x(t) + 3\frac{d^2}{dt^2}x(t)$

(b) $y(t) = e^{x(t)}$

(c) $y(t) = |Cosx(t)|$

(d) $y(t) = log|x(t)|$

(e) $y(t) = 2x(t) + 3x^2(t)$

(2) Test the following continuous time systems for time invariance.

(a) $y(t) = x(-t + 2)$

(b) $y(t) = 2x(t) + 3$

(3) Draw a simple RC high pass filter circuit and determine

(a) Transfer Function

(b) Magnitude Response

(c) Phase Response

(d) Impulse Response

(4) Use convolution method and obtain x(t) * x(t), where x(t)= A rect(t/T).

(5) Compute and plot the convoluted signal x(t) * y(t), given x(t) = u(t-1) – u(t-4), y(t) = e^{-2t}u(t)

(6) Obtain the convolution of the signals x(t) = u(t) and y(t) = u(t) - u(t-2)

13. Quiz Questions

(1) A physical device, that generates an output as response for a given input is called

(A) System

(B) Signal

(C) System Response

(D) Convolution

(2) For any system, the input and output characteristics do not change with time is called

(A) Linear system

(B) Time invariant system

(B) Time variant system

(C) Non linear system

(3) Which of the following system is dynamic

(A) $y(t) = x^2(t)$

(B) $y(t) = tx(t)$

(C) $y(t) = \frac{d}{dt}(x(t))$

(D) $y(t) = 2x(t) + 3$

(4) A system whose output depends on past and future inputs is called

(A) Static

(B) Dynamic

(C) Causal

(D) All

(5) A system is said to be linear if it satisfies

(A) Homogeneity

(B) Additive

(C) Superposition Principle

(D) All the above

(6) Which of the following system is time variant

(A) $y(t) = 2x(t)$

(B) $y(t) = x(2t)$

(C) $y(t) = \frac{d}{dt}(x(t))$

(D) $y(t) = x(t-2)$

(7) A system having differential equation $y(t) = x(t) + 2\frac{d}{dt}x(t) + 3\frac{d^2}{dt^2}x(t)$ is

(A) Linear

(B) Time invariant

(C) Causal

(D) All the above

(8) Which of the following system is causal

(A) y(t) = x(t – 2) + x(t + 4)

(B) y(t) = (t – 4) x(t + 1)

(C) y(t) = (t + 4) x(t – 1)

(D) y(t) = (t + 5) x(t + 5)

(9) The transfer function of a system y(t) = 2x(t) + 3x(t-1) is

(A) $\boldsymbol{H(\omega) = 2 + 3Cos\omega + j3Sin\omega}$

(B) $\boldsymbol{H(\omega) = 2 + 3Cos\omega - j3Sin\omega}$

(C) $H(\omega) = 2 + 3Sin\omega + j3Cos\omega$

(D) $H(\omega) = 2 + 3Sin\omega - j3Cos\omega$

(10) The transfer function of a low pass RC circuit is

(A) $\boldsymbol{H(\omega) = \frac{1}{1+j\omega RC}}$

(B) $\boldsymbol{H(\omega) = \frac{1}{1-j\omega RC}}$

(C) $H(\omega) = \frac{jwRC}{1+j\omega RC}$

(D) $H(\omega) = \frac{jwRC}{1-j\omega RC}$

(11) The range of frequencies for which the magnitude $|X(\omega)|$ remains within $\frac{1}{\sqrt{2}}$ of its maximum value is called

(A) Bandwidth

(B) Rise time

(C) Peak time

(D) Spectrum

(12) Response of LTI system is

(A) $\boldsymbol{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t+\tau)d\tau}$

(B) $\boldsymbol{y(t) = \int_{-\infty}^{\infty} x(t+\tau)h(t)dt}$

(C) $\boldsymbol{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$

(D) $y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)d\tau$

(13) The product of Bandwith and Rise time of a RC low pass filter is

(A) 0.35

(B) 0.75

(C) 0.95

(D) Depends on RC

(14) The characteristics of a distortion less system

(A) Magnitude Response is Constant

(B) Phase Response is Linear

(C) Bandwidth is Infinity

(D) All the above

(15) System y(t) = t x(t) is

(A) Linear Time Invariant

(B) Non-linear Time Invariant

(C) Linear Time Variant.

(D) Non-linear Time Variant

(16) The characteristics of a filter are

(A) Some components may be boosted in strength

(B) Some components may be attenuated

(C) Some components may be unchanged

(D) All the above

(17) Condition for a physical realizable system is

(A) $\int_{-\infty}^{\infty} |\boldsymbol{H}(\boldsymbol{\omega})|^2 \, \boldsymbol{d\omega} < \infty$

(B) $\int_{-\infty}^{\infty} \frac{|\boldsymbol{ln}|\boldsymbol{H}(\boldsymbol{\omega})||}{\boldsymbol{1+\omega^2}} \boldsymbol{d\omega} < \infty$

(C) Both (A) and (B)

(D) Neither (A) nor (B)

(18) Which of the following system is linear, time invariant, causal and BIBO stable.

(A) $y(t) = x^2(t)$

(B) $y(t) = x(2t)$

(C) $y(t) = \frac{d}{dt}(x(t))$

(D) $y(t) = x(t-2)$

(19) What is the convolution of two rectangular signals having same width?

(a) Trapezoidal signal

(b) Triangular signal

(c) Rectangular signal

(d) Sinc signal

(20) What is the convolution of two rectangular signals with unequal widths?

(a) Trapezoidal signal

(b) Triangular signal

(c) Rectangular signal

(d) Sinc signal